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On a discrete version of the CP^1 sigma model and surfaces immersed in $\mathbb{R}^3$

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Abstract
We present a discretization of the CP^1 sigma model. We show that the discrete CP^1 sigma model is described by a nonlinear partial second-order difference equation with rational nonlinearity. To derive discrete surfaces immersed in three-dimensional Euclidean space a 'complex' lattice is introduced. The so-obtained surfaces are characterized in terms of the quadrilateral cross-ratio of four surface points. In this way we prove that all surfaces associated with the discrete CP^1 sigma model are of constant mean curvature. An explicit example of such discrete surfaces is constructed.

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1. Introduction

In the last century, the problem of proper discretization of surfaces immersed in multi-dimensional spaces generated a great deal of interest and activity in several mathematical, physical, biological as well as numerical fields of research. For a comprehensive review of this subject see the book by A I Bobenko and R Seiler [1] and the references therein. In particular, discrete surfaces with constant mean curvature (CMC) have been shown to play an essential role in several applications to nonlinear phenomena in various areas of physics, such as two-dimensional gravity [2], string theory [3], quantum field theory [4], fluid dynamics [5] and theory of membranes [6, 7]. Our interest in this problem is partially motivated by these applications and by the development of new numerical computational tools in the studies of discrete surfaces immersed in three-dimensional Euclidean space $\mathbb{R}^3$.

The objective of this paper is to construct and investigate discrete CMC surfaces in $\mathbb{R}^3$, associated with completely integrable systems. Our starting point is the CP^1 sigma model equation which can be written as a compatibility condition for two linear spectral equations [8–10]. A discrete Lax representation, written in terms of a projection matrix, allows us to find a discrete version of the $\mathcal{C}P^1$ sigma model equation. To be able to associate with the Lax pair a position vector $\vec{X}$ in $\mathbb{R}^3$, we introduce a complex structure on the lattice. As a consequence, we show that we can act with the conjugation operation on functions defined on the lattice. Then the Lax pair for the discrete $\mathcal{C}P^1$ equation coincides with the compatibility condition which guarantees the existence of the position vector $\vec{X}$. This fact allows us to consider discrete surfaces immersed in Euclidean space $\mathbb{R}^3$ and to show that they are all discrete CMC surfaces.

In section 2, starting from the two-dimensional Euclidean $\mathcal{C}P^1$ sigma-model, we give a brief review of CMC surfaces in $\mathbb{R}^3$. In section 3 we introduce a discrete version of the $\mathcal{C}P^1$ sigma model equation. In section 4 we discuss the role of the lattice in the definition of the discrete $\mathcal{C}P^1$ equation. We introduce a 'complex' lattice structure and present some of its properties. Section 5 contains examples of solutions of the discrete $\mathcal{C}P^1$ sigma model, in both the real and complex domains. Section 6 contains the representation of its associated discrete surfaces and its geometric characterization, while in section 7 we provide an example. Section 8 contains our conclusions. A few notions, definitions, relations and some theorems dealing with discrete systems of difference equations are presented in appendices A, B and C.

2. The $\mathcal{C}P^1$ representation for surfaces immersed in three-dimensional Euclidean space

The starting point of this calculation is a two-dimensional Euclidean $\mathcal{C}P^1$ sigma model with internal symmetry group $O(3)$. In the stereographic projection representation this model is defined by the Lagrangian [8]

$$\mathcal{L} = \int \frac{|\partial w|^2 + |\partial \bar{w}|^2}{1+|w|^2}\, dz\, d\bar{z}$$

where w is a complex field of the complex variables z and its conjugate $\bar{z}$. By the symbols ∂ and $\bar{\partial}$ we denote the partial derivative with respect to z and $\bar{z}$ respectively. The corresponding Euler–Lagrange equation reads

$$\partial\bar{\partial}w - \frac{2\bar{w}}{1+|w|^2}\partial w \bar{\partial} w = 0. \tag{2.1}$$

The $\mathcal{C}P^1$ equation (2.1) can be obtained as the compatibility condition of two linear equations for an auxiliary complex matrix function Ψ [9]

$$\partial\Psi = \frac{2}{1+\lambda}[\partial P, P]\Psi \qquad \bar{\partial}\Psi = \frac{2}{1-\lambda}[\bar{\partial} P, P]\Psi \tag{2.2}$$

where the brackets [,] denote the commutator, λ is a complex spectral parameter and P is a Hermitian 2×2 projector matrix whose unique representation in terms of w is given by

$$P = (1+|w|^2)^{-1}\begin{pmatrix} 1 & \bar{w} \\ w & |w|^2 \end{pmatrix}. \tag{2.3}$$

The compatibility condition of equations (2.2) is equivalent to the $\mathcal{C}P^1$ equation (2.1). They can be written in the conservation law form

$$\partial K + \bar{\partial} M = 0 \tag{2.4}$$

where K and M are 2×2 traceless matrices of the form

$$\begin{aligned} K &= [\bar{\partial}P, P] \\ &= (1+|w|^2)^{-1}\begin{pmatrix} \bar{w}\bar{\partial}w - w\bar{\partial}\bar{w}, \bar{\partial}\bar{w} + \bar{w}^2\bar{\partial}w \\ -\bar{\partial}w - w^2\bar{\partial}\bar{w}, w\bar{\partial}\bar{w} - \bar{w}\bar{\partial}w \end{pmatrix} \end{aligned} \tag{2.5}$$

$$M = [\partial P, P] = -K^{\dagger}. \tag{2.6}$$

Let us recall that any anti-Hermitian matrix X of rank 2

$$X^{\dagger} = -X \tag{2.7}$$

can be mapped onto a vector in $\mathbb{R}^3$

$$\vec{X} = (X_1, X_2, X_3) : \mathcal{D} \to \mathbb{R}^3 \tag{2.8}$$

by setting

$$X = \mathrm{i} X_j \sigma_j \tag{2.9}$$

where σ_j are the usual Pauli matrices which satisfy the following relations:

$$\sigma_i \sigma_j = \mathrm{i}\varepsilon_{ijk}\sigma_k + \delta_{ij} \qquad \sigma_j = \sigma_j^{\dagger}.$$

In the chosen local coordinates z and $\bar{z}$, the vector $\vec{X}$ describes a surface. The tangents to the surface described by $\vec{X}$ are ∂X and $\bar{\partial} X$. The complex structure in the coordinate space and the mapping $\vec{X} \leftrightarrow X$ provide the relation

$$(\partial X)^{\dagger} = -\bar{\partial} X. \tag{2.10}$$

We can relate the tangent vectors ∂X and $\bar{\partial} X$ to the projector P through the following formulae:

$$\partial X = 2\mathrm{i}[\partial P, P] = 2\mathrm{i}M = -2\mathrm{i}K^{\dagger} \qquad \bar{\partial} X = -2\mathrm{i}[\bar{\partial} P, P] = -2\mathrm{i}K. \tag{2.11}$$

The compatibility condition $\partial\bar{\partial} X = \bar{\partial}\partial X$ for (2.11) implies equation (2.4), i.e.

$$\partial[\bar{\partial} P, P] + \bar{\partial}[\partial P, P] = 0 \tag{2.12}$$

which coincides with the result obtained from the compatibility condition for the Lax pair (2.2). We can interpret the tangent vectors ∂X and $\bar{\partial} X$ as the coefficients of a closed (and exact) matrix 1-form, which we can integrate along an arbitrary path γ in the complex plane.

The tangent vectors ∂X and $\bar{\partial} X$ can be decomposed in terms of Pauli matrices σ_j and written in terms of $w(z, \bar{z})$

$$\begin{aligned}
\partial X = 2\mathrm{i}[\partial P, P] &= \frac{-2\mathrm{i}}{(1+|w|^2)^2}\left\{(w\partial\bar{w} - \bar{w}\partial w)\sigma_3 + \frac{1}{2}(\partial w - \partial\bar{w} + w^2\partial\bar{w} - \bar{w}^2\partial w)\sigma_1\right. \\
&\quad \left. + \frac{1}{2\mathrm{i}}(\partial w + \partial\bar{w} + w^2\partial\bar{w} + \bar{w}^2\partial w)\sigma_2\right\} \\
\bar{\partial} X = -2\mathrm{i}[\bar{\partial} P, P] &= \frac{2\mathrm{i}}{(1+|w|^2)^2}\left\{(w\bar{\partial}\bar{w} - \bar{w}\bar{\partial} w)\sigma_3\right. \\
&\quad \left. + \frac{1}{2}(\bar{\partial} w - \bar{\partial}\bar{w} + w^2\bar{\partial}\bar{w} - \bar{w}^2\bar{\partial} w)\sigma_1 - \frac{1}{2\mathrm{i}}(\bar{\partial} w + \bar{\partial}\bar{w} + w^2\bar{\partial}\bar{w} + \bar{w}^2\bar{\partial} w)\sigma_2\right\}.
\end{aligned} \tag{2.13}$$

Equations (2.13) allow us to calculate the position vector $\vec{X}$, whose components are

$$\begin{aligned}
X_1 &= \mathrm{i}\int_{\gamma} \frac{1}{(1+|w|^2)^2}\{[(1+\bar{w}^2)\partial w - (1+w^2)\partial\bar{w}]\,\mathrm{d}z - [(1+w^2)\bar{\partial}\bar{w} - (1+\bar{w}^2)\bar{\partial} w]\,\mathrm{d}\bar{z}\} \\
X_2 &= \int_{\gamma} \frac{1}{(1+|w|^2)^2}\{[(1-\bar{w}^2)\partial w + (1-w^2)\partial\bar{w}]\,\mathrm{d}z + [(1-w^2)\bar{\partial}\bar{w} + (1-\bar{w}^2)\bar{\partial} w]\,\mathrm{d}\bar{z}\} \\
X_3 &= -2\int_{\gamma} \frac{1}{(1+|w|^2)^2}\{[\bar{w}\partial w + \omega\partial\bar{w}]\,\mathrm{d}z + [w\bar{\partial}\bar{w} + \bar{w}\bar{\partial} w]\,\mathrm{d}\bar{z}\}.
\end{aligned} \tag{2.14}$$

Moreover, the unit normal vector to the surface reads

$$N = -\frac{1-|w|^2}{1+|w|^2}\sigma_3 + \frac{w+\bar{w}}{1+|w|^2}\sigma_1 - \mathrm{i}\frac{w-\bar{w}}{1+|w|^2}\sigma_2. \qquad (2.15)$$

Defining

$$p^2 = \frac{(\partial w \bar{\partial} \bar{w})^{\frac{1}{2}}}{1+|w|^2} \qquad (2.16)$$

the Gauss curvature and the mean curvature take the form

$$K = -p^{-2}\partial\bar{\partial}\ln p \qquad H = 1. \qquad (2.17)$$

The solutions of $\mathcal{C}P^1$ sigma model (2.1) determine an immersion of a CMC surface in $\mathbb{R}^3$. A particular important class of solutions of $\mathcal{C}P^1$ sigma model (2.1) is given by arbitrary holomorphic function $w(z)$ or antiholomorphic function $w(\bar{z})$.

3. The discrete $\mathcal{C}P^1$ sigma model equation

In this section we construct and investigate a discrete version of the $\mathcal{C}P^1$ sigma model equation (2.1) by using a discretized version of its linear spectral problem (2.2).

Let us consider two linear discrete equations for an auxiliary complex matrix function $\psi_{n,m} \in \mathcal{C}^2$

$$\psi_{n+1,m} = \left[1 + \frac{2}{1+\lambda}M_{n,m}\right]\psi_{n,m} \qquad \psi_{n,m+1} = \left[1 + \frac{2}{1-\lambda}K_{n,m}\right]\psi_{n,m} \qquad (3.1)$$

where $M_{n,m}$ and $K_{n,m}$ are 2×2 complex-valued matrices defined on the lattice of the integer variables (n, m), and $\lambda \in \mathcal{C}$ represents the spectral parameter. For any parameter λ the equations of motion are obtained from the compatibility conditions for (3.1) and read

$$\Delta_n K_{n,m} = M_{n,m+1}K_{n,m} - K_{n+1,m}M_{n,m} \qquad \Delta_m M_{n,m} = K_{n+1,m}M_{n,m} - M_{n,m+1}K_{n,m} \qquad (3.2)$$

where

$$\Delta_n f_{n,m} \doteq f_{n+1,m} - f_{n,m} \qquad \Delta_m f_{n,m} \doteq f_{n,m+1} - f_{n,m} \qquad (3.3)$$

are difference operators acting on the functions $f_{n,m}$ defined on the lattice. As an alternative description we introduce the commuting shift operators S and T such that

$$T f_{n,m} = f_{n,m+1} \qquad S f_{n,m} = f_{n+1,m} \qquad [S, T] = 0. \qquad (3.4)$$

In terms of these definitions the operators Δ_n and Δ_m can be rewritten as

$$\Delta_n \equiv \Delta_1 = S - 1 \qquad \Delta_m \equiv \Delta_2 = T - 1 \qquad (3.5)$$

and equations (3.2) become

$$\Delta_1 K + \Delta_2 M = 0 \qquad (3.6)$$

$$\Delta_1 K = (TM)K - (SK)M \qquad (3.7)$$

where K stands for $K_{n,m}$ and M for $M_{n,m}$. From now on, whenever possible, for the sake of convenience we shall suppress the arguments n, m in all functions in order to have a simpler notation. As in the continuous case, we represent the matrices K and M in terms of a projector matrix P such that

$$P^2 = P \qquad (TP)^2 = TP \qquad (SP)^2 = SP \qquad (STP)^2 = STP. \qquad (3.8)$$

Since the difference operators satisfy the deformed Leibnitz rule (A.1), to obtain the explicit expressions of K and M in terms of P, we look for a representation of those matrices in terms of powers of the shift operators

$$\begin{aligned} K &= [T^{\alpha_0} S^{\beta_0} \Delta_2 P]P - (T^{\alpha_1} S^{\beta_1} P)(T^{\alpha_2} S^{\beta_2} \Delta_2 P) \\ M &= [T^{\gamma_0} S^{\delta_0} \Delta_1 P]P - (T^{\gamma_1} S^{\delta_1} P)(T^{\gamma_2} S^{\delta_2} \Delta_1 P) \end{aligned} \tag{3.9}$$

where the exponents $\alpha_i, \beta_i, \gamma_i, \delta_i, i = 0, 1, 2$ are integers. We determine them by requiring that equations (3.6) and (3.7) reduce to one and only one equation for the projector P (for more details see appendix B). We get

$$K = 2(TP)P - TP - P \qquad M = 2(SP)P - SP - P. \tag{3.10}$$

The compatibility condition (3.7) is reduced to a difference equation for the projector P

$$(\Delta_2\Delta_1 P)P - (TSP)\Delta_2\Delta_1 P = 0 \tag{3.11}$$

or equivalently,

$$(TSP)(1 - TP - SP) + (TP + SP)P - P = 0. \tag{3.12}$$

If P is a Hermitian projection matrix then one can represent it in a unique way in terms of a complex function $w_{n,m}$ as in equation (2.3)

$$P_{n,m} = \frac{1}{1 + |w_{n,m}|^2} \begin{pmatrix} 1 & \bar{w}_{nm} \\ w_{nm} & |w_{nm}|^2 \end{pmatrix} \qquad P^\dagger_{n,m} = P_{n,m} \tag{3.13}$$

where

$$|w_{n,m}|^2 = w_{n,m}\bar{w}_{n,m}. \tag{3.14}$$

Eliminating the projector $P_{n,m}$ from the compatibility condition (3.12) we obtain the discrete version of $\mathcal{C}P^1$ sigma model equation for the complex function $w_{n,m}$

$$\begin{aligned} \Delta_1\Delta_2 w_{n,m} = {} & \{(\Delta_1 w_{n,m})[(\Delta_2 w_{n+1,m})\bar{w}_{n+1,m}|w_{n,m+1}|^2 + (\Delta_2 w_{n,m})(|w_{n+1,m}|^2\bar{w}_{n,m+1} + \bar{w}_{n,m+1} \\ & + \bar{w}_{n+1,m})]\}\{1 + |w_{n+1,m}|^2 w_{n,m}\bar{w}_{n,m+1} + w_{n,m}(\bar{w}_{n+1,m} + \bar{w}_{n,m+1})\}^{-1}. \end{aligned} \tag{3.15}$$

Equation (3.15) is a nonlinear second-order difference equation with rational nonlinearity. The presence of rational nonlinearity seems to be a common feature of integrable nonlinear difference equations. The same phenomenon appears in the case of the discrete Riccati equations [11].

4. The 'complex' lattice structure

Equation (3.15) is the result of the compatibility condition for equations (3.1). We obtained it by purely algebraic methods, using just the definition of the shift operators S and T (3.4). No property of the lattice is involved.

To be able to provide a complete discrete analogue of the $\mathcal{C}P^1$ sigma model and of the CMC surface in $\mathbb{R}^3$ we need to define complex variables on the lattice, one the complex conjugate of the other. To do so, let us introduce shift operators on a two-dimensional lattice of variables z_1 and z_2 and define the properties of the shift operators and the lattice in such a way as to provide a discrete analogue of a complex plane.

Let us consider two shift operators, say Λ_{z_1} and Λ_{z_2}, which act on a complex function $w(z_1, z_2)$, where z_1 and z_2 are two independent complex variables

$$\Lambda_{z_1} w(z_1, z_2) \doteq w(z_1 + \sigma_1, z_2) \qquad \Lambda_{z_2} w(z_1, z_2) \doteq w(z_1, z_2 + \sigma_2). \tag{4.1}$$

Here σ_1 and σ_2 are the lattice spacings in the independent variables z_1 and z_2, respectively. The variables z_1 and z_2 (and consequently the spacings σ_1 and σ_2) can be defined in two different ways having different implications for the equation in $w(z_1, z_2)$ (for example equation (3.15)).

1. The independent variables z_1 and z_2 are real and represent two real axes and σ_1 and σ_2 are their real lattice spacings. In this case the two shift operators (4.1) are linearly independent and the complex conjugation will imply the complex conjugation of the function $w(z_1, z_2)$, i.e. $\overline{w(z_1, z_2)} = \bar{w}(z_1, z_2)$.
2. The independent variables z_1 and z_2 are complex. They represent two independent axes and σ_1 and σ_2 are the complex lattice spacings on them (which, in all generality, we can set to be one the complex conjugate of the other $\bar{\sigma}_1 = \sigma_2$). Then, if we can set

$$\bar{z}_1 = z_2 \tag{4.2}$$

we can call $z_1 = z$ and $\sigma_1 = \sigma$. So we have

$$\Lambda_z w(z, \bar{z}) = w(z + \sigma, \bar{z}) \qquad \Lambda_{\bar{z}} w(z, \bar{z}) = w(z, \bar{z} + \bar{\sigma}) \tag{4.3}$$

and see immediately that $\overline{\Lambda_z} = \Lambda_{\bar{z}}$, i.e. Λ_z and $\Lambda_{\bar{z}}$ are the complex conjugates of each other.

We would like the two variables z_1 and z_2 to depend explicitly on two indices, say n and m, which define the position of a point on the complex plane $\mathcal{C}$. Let us select them by setting

$$z_1 = O_1 + n\sigma_1 \qquad z_2 = O_2 + m\sigma_2 \tag{4.4}$$

where (O_1, O_2) can always, with no loss of generality, be set equal to (0, 0).

Definition (4.4) is compatible with case 1. In case 2, however, we have $\bar{z}_1 \neq z_2$, for $n \neq m$. Moreover, we have

$$\begin{aligned} \overline{\Lambda_{z_1} w(n\sigma, m\bar{\sigma})} &= \bar{w}(m\sigma, n\bar{\sigma} + \bar{\sigma}) = \overline{\Lambda_{z_1}} \bar{w}(m\sigma, n\bar{\sigma}) = \Lambda_{z_2} \bar{w}(m\sigma, n\bar{\sigma}) \\ \overline{\Lambda_{z_2} w(n\sigma, m\bar{\sigma})} &= \bar{w}(m\sigma + \sigma, n\bar{\sigma}) = \overline{\Lambda_{z_2}} \bar{w}(m\sigma, n\bar{\sigma}) = \Lambda_{z_1} \bar{w}(m\sigma, n\bar{\sigma}) \end{aligned} \tag{4.5}$$

i.e. $\overline{\Lambda_{z_1}} = \Lambda_{z_2}$ when $\sigma_2 = \bar{\sigma}$.

To ensure that $\bar{z}_1 = z_2$, we could set

$$z_1 = n\sigma + m\bar{\sigma} \qquad z_2 = n\bar{\sigma} + m\sigma. \tag{4.6}$$

However, in this case equation (4.1) is not satisfied.

Up to now we have not been able to find a representation of z_1 and z_2 in terms of two integer indices (n, m) such that both equations (4.1), (4.2) are satisfied. So in the following we will consider z_1 and z_2 given by equation (4.4) and either, at a first instance, σ_1 and σ_2 real or, in the second case, σ_1 and σ_2 being the complex conjugates of each other.

Let us analyse in more detail the complex structure of the lattice proposed in case 2. In this case the independent discrete indices (n, m) of the lattice are identified with the ordered pair of complex variables $(z, \bar{z}) \in \mathcal{C}$. These variables provide two independent directions on the complex plane $\mathcal{C}$, characterized by a complex constant σ (which we call an *elementary complex number*), such that

$$z_n = n\sigma \qquad \bar{z}_m = m\bar{\sigma}. \tag{4.7}$$

Let us note that z_n and $\bar{z}_m$ are the complex conjugates of each other only when $m = n$. Moreover

$$\sigma = \sigma_R + \mathrm{i}\sigma_I \tag{4.8}$$

where σ_R and σ_I are fixed real numbers. The continuous limit of z and $\bar{z}$ is defined by $\sigma \to 0$ (i.e. $\sigma_R \to 0$, $\sigma_I \to 0$) and $n, m \to \infty$, in such a way that z and $\bar{z}$ are finite. A function $w(z_n, \bar{z}_m)$ can be identified with a function of the integer numbers n, m by assuming

$$w(z_n, \bar{z}_m) = w(n\sigma, m\bar{\sigma}) = w_{n,m} \tag{4.9}$$

where n and m are respectively the coefficients of σ and $\bar{\sigma}$ in the definition of z and $\bar{z}$. We can now define the operation of complex conjugation on the functions defined on this lattice. For a function $w_{n,m}$ we have

$$\overline{w_{n,m}} = \overline{w(n\sigma, m\bar{\sigma})} = \bar{w}(n\bar{\sigma}, m\sigma) = \bar{w}(m\sigma, n\bar{\sigma}) = \bar{w}_{m,n}. \tag{4.10}$$

So the complex conjugate of a function $w_{n,m}$ interchanges the orders of the numbers n and m. A similar situation occurs for any matrix-valued function $A_{n,m}$ on the lattice. To obtain its Hermitian conjugate, $A^{\dagger}_{n,m}$, we need to transpose the matrix (in the matrix sense) and then to transform 'globally' in the above sense all its entries.

For brevity in the following we may also use the operators Re and Im defined by

$$\mathrm{Re}(w_{n,m}) = \frac{1}{2}(w_{n,m} + \overline{(w_{n,m})}) \qquad \mathrm{Im}(w_{n,m}) = \frac{1}{2\mathrm{i}}(w_{n,m} - \overline{(w_{n,m})}). \tag{4.11}$$

Re $(w_{n,m})$ and Im $(w_{n,m})$ are both real functions.

The complex structure of the lattice reflects on the operators defined on it. In particular, for the shift operators defined in equation (3.4), we have

$$Sw_{n,m} = w((n+1)\sigma, m\bar{\sigma}) = w_{n+1,m} \qquad Tw_{n,m} = w(n\sigma, (m+1)\bar{\sigma}) = w_{n,m+1} \tag{4.12}$$

where the operator S shifts z by one in the positive direction, while T acts in the same way on $\bar{z}$. Then one gets (see also (4.5))

$$\begin{aligned} \overline{(Sw_{n,m})} &= \overline{(w_{n+1,m})} = (\bar{w}_{m,n+1}) = (T\bar{w}_{m,n}) \\ \overline{(Tw_{n,m})} &= \overline{(w_{n,m+1})} = (\bar{w}_{m+1,n}) = (S\bar{w}_{m,n}). \end{aligned} \tag{4.13}$$

So for the difference operators defined by (3.5) one has

$$\begin{aligned} \overline{(\Delta_1 w)} &= \overline{((S-1)w)} = (T-1)\bar{w} = \Delta_2\bar{w} \\ \overline{(\Delta_2 w)} &= \overline{((T-1)w)} = (S-1)\bar{w} = \Delta_1\bar{w}. \end{aligned} \tag{4.14}$$

Let us note that as $\sigma \to 0$, the difference operators $\frac{\Delta_1}{\sigma}$ and $\frac{\Delta_2}{\bar{\sigma}}$ tend to ∂_z and $\partial_{\bar{z}}$, respectively.

As an example, let us consider the particular function

$$w_{n,m} = n\sigma \tag{4.15}$$

defined over the lattice, which in terms of z would be $w(z, \bar{z}) = z$. Its complex conjugate takes the value

$$\overline{w_{m,n}} = \bar{w}_{m,n} = n\bar{\sigma}.$$

In this case the operations on the integer indices n, m, which correspond to Re and Im operators on complex numbers, lead to

$$\mathrm{Re}(w_{n,m}) = n\sigma_R \qquad \mathrm{Im}(w_{n,m}) = n\sigma_I \tag{4.16}$$

and $w_{n,m}|_{n=0} = 0$.

5. Solutions of the discrete $\mathcal{C}P^1$ sigma model equation

At this point we proceed to construct several classes of solutions of the discrete $\mathcal{C}P^1$ equation (3.15).

1. The simplest class of solutions of (3.15) is the translationally invariant solutions in z_1 or z_2 directions

$$w_{n,m} = v_n \qquad \text{or} \qquad w_{n,m} = u_m. \tag{5.1}$$

It can immediately be proved that these solutions satisfy equation (3.15) identically. Let us note that in a continuous case the solutions (5.1) correspond to holomorphic and antiholomorphic solutions of (2.1), respectively.

2. We now look for the unimodular solutions of (3.15), i.e. solutions for which the following condition

$$|w_{n,m}|^2 = 1 \tag{5.2}$$

holds. In this case equations (3.15) are reduced to a second-order difference equation with the cubic nonlinearity

$$w_{n,m+1} + w_{n+1,m} - w_{n,m}w_{n+1,m+1}(\bar{w}_{n+1,m} + \bar{w}_{n,m+1}) = 0 \tag{5.3}$$

which, by virtue of equation (5.2), admits a simple rational representation

$$w_{n+1,m+1} = \frac{w_{n,m+1}w_{n+1,m}}{w_{n,m}}. \tag{5.4}$$

Using the operators Δ_2, Δ_1, S and T, we can rewrite equation (5.4) as

$$\Delta_2 \frac{Sw}{w} = 0 \qquad \text{or} \qquad \Delta_1 \frac{Tw}{w} = 0. \tag{5.5}$$

Consequently, for arbitrary complex functions u_n and v_m, the expressions

$$\frac{Sw}{w} = u(z) \qquad \frac{Tw}{w} = v(\bar{z}) \tag{5.6}$$

are translational invariant solutions of (5.5) in the σ_2 and σ_1 directions, respectively. Then, a unimodular solution of (3.15) can be written as

$$w_{n,m} = a_m \prod_{j=0}^{n-1} u_j \qquad |a_m|^2 = 1 \qquad |u_n|^2 = 1 \qquad a_m \in \mathcal{C}. \tag{5.7}$$

Assuming σ_1 and σ_2 to be real, then the class of unimodular solutions of (5.2) takes the form

$$w_{n,m} = \mathrm{e}^{\mathrm{i}(\xi_m + \varphi_n)} \qquad \xi_m, \varphi_n \in \mathcal{R}. \tag{5.8}$$

If σ_1 and σ_2 are complex and one is the complex conjugate of the other, the situation is more complicated; equations (5.6) are still valid but condition (5.2) has no nontrivial solution. Equation (5.8) is no longer a solution of equation (5.3).

3. In general we can rewrite equation (3.15) as

$$\begin{aligned} w_{n+1,m+1} = \{&w_{n,m+1}(1 + |w_{n+1,m}|^2) + w_{n+1,m}(1 + |w_{n,m+1}|^2) \\ &- w_{n,m}(1 - |w_{n,m+1}|^2|w_{n+1,m}|^2)\}\{1 - |w_{n,m+1}|^2|w_{n+1,m}|^2 \\ &+ w_{n,m}[\bar{w}_{n+1,m}(1 + |w_{n+m+1}|^2) + \bar{w}_{n,m+1}(1 + |w_{n+1,m}|^2)]\}^{-1}. \end{aligned} \tag{5.9}$$

This form of the discrete $\mathcal{C}P^1$ sigma model equation implies that the initial data have to be imposed on both lattice axes z_1 and z_2 (see figure 1). If this takes place and, in addition, the denominator of equation (5.9) is different from zero then the solution is uniquely recursively determined. This is an analogue of the Goursat–Darboux boundary value problem for a second-order hyperbolic equation in the continuous case [12].

We end this section by considering the continuous limit of equation (3.15). To do so we assume that the real parameters σ_1 and σ_2 in case 1 or the complex parameter σ in case 2 are small and vanishing, in such a way, however, as to leave z_1, z_2 and z finite. This occurs whenever σ_1, σ_2 and σ tend to zero and n and m tend to infinity. Then, taking into consideration equation (4.9), we get for case 1

$$\begin{aligned} w_{n+1,m} &= w(z_1 + \sigma_1, z_2) = w + \sigma_1 w_{z_1} + \tfrac{1}{2}\sigma_1^2 w_{z_1z_1} + \cdots \\ w_{n,m+1} &= w(z_1, z_2 + \sigma_2) = w + \sigma_2 w_{z_2} + \tfrac{1}{2}\sigma_2^2 w_{z_2z_2} + \cdots \\ w_{n+1,m+1} &= w(z_1 + \sigma_1, z_2 + \sigma_2) = w + \sigma_1 w_{z_1} + \sigma_2 w_{z_2} \\ &\quad + \tfrac{1}{2}\left[\sigma_1^2 w_{z_1z_1} + \sigma_2^2 w_{z_2z_2} + 2\sigma_1\sigma_2 w_{z_2z_1}\right] + \cdots. \end{aligned} \tag{5.10}$$

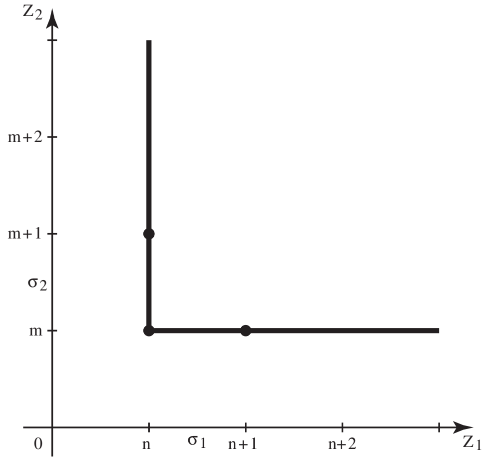

Figure 1. Initial data for equation (5.9).

The corresponding equations for case 2 are obtained from equation (5.10) by replacing σ_1 by σ, σ_2 by $\bar{\sigma}$, z_1 by z and z_2 by $\bar{z}$.

Introducing equation (5.10) into equation (5.9) and taking into account the corresponding different definitions of $\bar{w}_{n,m}$, in both cases we get equation (2.1) at the lowest order.

6. Representation of discrete surfaces in $\mathbb{R}^3$

The aim of this section is to derive a discrete version of the representation of a surface immersed in $\mathbb{R}^3$. In other words, we look for a discrete analogue of expressions (2.14) and (2.13). This is only possible in case 2, i.e. when $w_{n,m} = w(z_n, \bar{z}_m)$ and σ is a fixed complex number.

We start our analysis by defining the position vector $X = X_{n,m}$ on the lattice. In this derivation the complex structure of the lattice is essential in order to get a nontrivial result.

Let us assume that the following equation is valid

$$\Delta_2 X = K - (M)^\dagger = [(K)^\dagger - M]^\dagger \qquad \Delta_1 X = (K)^\dagger - M. \tag{6.1}$$

The compatibility condition for the position vector *X* is

$$\Delta_1(K - M^\dagger) + \Delta_2(M - K^\dagger) = 0. \tag{6.2}$$

In the continuous case we have proved that compatibility condition (2.4) for the Lax pair (2.2) coincides with the compatibility condition (2.12) for the position vector *X*. We show now that a similar situation takes place also in the discrete case. In view of equation (3.11), the compatibility condition for the discrete Lax pair (3.1) can be written as

$$[\Delta_1\Delta_2 P, P] - (\Delta_2 P + \Delta_1 P)\Delta_1\Delta_2 P - (\Delta_1\Delta_2 P)^2 = 0. \tag{6.3}$$

On the other hand, using definitions (3.10), we can rewrite the compatibility condition (6.2) for the position vector *X* in terms of the projector *P*

$$\Delta_1[\Delta_2 P, P] + \Delta_2[\Delta_1 P, P] = 0. \tag{6.4}$$

It can be shown that the difference equations (6.3) and (6.4) coincide. In fact, taking the Hermitian conjugation of equation (6.3) and adding or subtracting it from equation (6.3), we get

$$2[\Delta_1\Delta_2 P, P] - [\Delta_2 P + \Delta_1 P, \Delta_1\Delta_2 P] = 0 \tag{6.5}$$

$$\{\Delta_1\Delta_2 P, \Delta_1 P + \Delta_2 P\} + 2(\Delta_1\Delta_2 P)^2 = 0 \tag{6.6}$$

where the bracket {} denotes the anticommutator. On the other hand, expanding the expression (6.4) we get equation (6.5). The difference equation (6.6) can also be written as

$$\Delta_1(\Delta_2 P)^2 + \Delta_2(\Delta_1 P)^2 = 0 \tag{6.7}$$

which is identically satisfied, whenever the discrete $\mathcal{C}P^1$ equation (3.12) holds. So, we have proved that the compatibility condition (6.2) for the position vector $\vec{X}$ does not imply any additional constraints on the projector P. By using the relations (3.10) the matrices $K - M^\dagger$ and $M - K^\dagger$ can be simply expressed in terms of the projector P as

$$K - M^\dagger = 2[TP, P] \qquad M - K^\dagger = -2[SP, P]. \tag{6.8}$$

Since these matrices are expressed through commutators, they are traceless. This fact can be easily checked by using the decomposition of the projector P in terms of the Pauli matrices σ_j

$$P = \frac{1}{2}\sigma_0 + \frac{1}{2}\frac{(w+\bar{w})}{1+|w|^2}\sigma_1 + \frac{\mathrm{i}}{2}\frac{(\bar{w}-w)}{1+|w|^2}\sigma_2 + \frac{(1-|w|^2)}{2(1+|w|^2)}\sigma_3. \tag{6.9}$$

As a consequence of (6.1) and of the complex structure of the lattice, the matrix representation of the position vector $\vec{X}$ satisfies the reality condition $X^\dagger = X$. Thus we can express X in terms of the Pauli matrices σ_i and identify its coefficients with the components of a vector in three-dimensional Euclidean space $\mathbb{R}^3$

$$X = X^{(i)}\sigma_i \leftrightarrow \vec{X} = (X^{(1)}, X^{(2)}, X^{(3)}) \in \mathbb{R}^3. \tag{6.10}$$

Hence, if the matrix $X(z, \bar{z})$ obeys conditions (6.1), then we can treat the functions $X^{(a)}(z, \bar{z}), a = 1, 2, 3$, implicitly defined in (6.10), as the discrete coordinates of a surface defined on the lattice in $\mathbb{R}^3$. System (6.1) can be considered a discrete variant of the representation (2.13). Note that after the substitution of equation (6.8) into (6.1) we get

$$\Delta_2 X = 2[TP, P] \qquad \Delta_1 X = -2[SP, P]. \tag{6.11}$$

Equations (6.11) in the continuous limit tend to the original representation (2.11) up to an i factor as $X_{n,m} \in \mathbb{R}$ while $X \in \mathbb{I}$.

The general solution of the difference equations (6.1) (derived in appendix C) has the form (see figure 2)

$$X_{n,m} = A_0 - \sum_{n'=0}^{n-1}\left(M_{n',0} - K^\dagger_{n',0}\right) + \sum_{m'=0}^{m-1}\left(K_{n,m'} - M^\dagger_{n,m'}\right) \qquad A_0 \in \mathcal{R}. \tag{6.12}$$

The real constant matrix A_0 can be decomposed in terms of the Pauli matrices

$$A_0 = a_l\sigma_l \tag{6.13}$$

as, due to equation (6.11), $X_{n,m}$ is traceless. Since A_0 is a constant we can set it to zero without loss of generality. Equation (6.12) can be expressed equivalently, using equation (6.8), in terms of the projector P

$$X_{n,m} = -4\mathrm{i}\sum_{n'=0}^{n-1} P^{(j)}_{n'+1,0}P^{(k)}_{n',0}\epsilon_{jkl}\sigma_l + 4\mathrm{i}\sum_{m'=0}^{m-1} P^{(j)}_{n,m'+1}P^{(k)}_{n,m'}\epsilon_{jkl}\sigma_l. \tag{6.14}$$

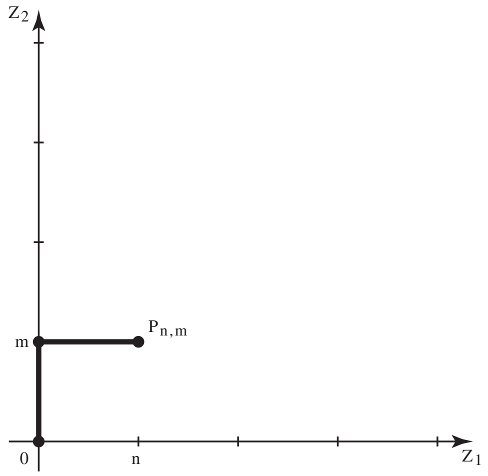

Figure 2. Integration path of equation (6.1) used to obtain equation (6.12).

Using equation (6.9) we can write the components of the position vector $X^{(l)}_{n,m}$ in the form

$$\begin{aligned} X^{(1)}_{n,m} &= -4\sum_{n'=0}^{n-1}\frac{\mathrm{Im}(w_{n'+1,0})(1-|w_{n',0}|^2)-(1-|w_{n'+1,0}|^2)\,\mathrm{Im}(w_{n',0})}{(1+|w_{n'+1,0}|^2)(1+|w_{n',0}|^2)} \\ &\quad+\sum_{m'=0}^{m-1}\frac{\mathrm{Im}(w_{n,m'+1})(1-|w_{n,m'}|^2)-(1-|w_{n,m'+1}|^2)\,\mathrm{Im}(w_{n,m'})}{(1+|w_{n,m'+1}|^2)(1+|w_{n,m'}|^2)} \\ X^{(2)}_{n,m} &= 4\sum_{n'=0}^{n-1}\frac{\mathrm{Re}(w_{n'+1,0})(1-|w_{n',0}|^2)-(1-|w_{n'+1,0}|^2)\mathrm{Re}(w_{n',0})}{(1+|w_{n'+1,0}|^2)(1+|w_{n',0}|^2)} \\ &\quad-\sum_{m'=0}^{m-1}\frac{\mathrm{Re}(w_{n,m'+1})(1-|w_{n,m'}|^2)-(1-|w_{n,m'+1}|^2)\mathrm{Re}(w_{n,m'})}{(1+|w_{n,m'+1}|^2)(1+|w_{n,m'}|^2)} \\ X^{(3)}_{n,m} &= -4\sum_{n'=0}^{n-1}\frac{\mathrm{Re}(w_{n'+1,0})\mathrm{Im}(w_{n',0})-\mathrm{Re}(w_{n',0})\,\mathrm{Im}(w_{n'+1,0})}{(1+|w_{n'+1,0}|^2)(1+|w_{n',0}|^2)} \\ &\quad+4\sum_{m'=0}^{m-1}\frac{\mathrm{Re}(w_{n,m'+1})\mathrm{Im}(w_{n,m'})-\mathrm{Im}(w_{n,m'+1})\mathrm{Re}(w_{n,m'})}{(1+|w_{n,m'+1}|^2)(1+|w_{n,m'}|^2)}. \end{aligned} \tag{6.15}$$

So we have $X^{(l)}_{n,m} \in \mathbb{R}^1 (l = 1, 2, 3)$ and can treat the functions $X^{(l)}_{n,m}$ as the coordinates on a discrete lattice of a surface in $\mathbb{R}^3$.

For the position vector $\vec{X}$ we can calculate the corresponding discrete tangent vectors and find their continuous limit. From equations (6.9), (6.11) we get the components of these tangent vectors by decomposing $\Delta_1 X$ and $\Delta_2 X$ in terms of the Pauli matrices σ_i

$$\Delta_1 X = \Delta_1 X^{(i)}\sigma_i \qquad \Delta_2 X = \Delta_2 X^{(i)}\sigma_i. \tag{6.16}$$

They are

$$\Delta_1 X^{(1)} = \frac{\mathrm{i}}{(1+|w_{n,m}|^2)(1+|w_{n+1,m}|^2)}[1+\bar{w}_{n,m}(w_{n+1,m}\Delta_n w_{n,m} - \bar{w}_{n+1,m}\Delta_n w_{n,m})]$$
$$\Delta_1 X^{(2)} = \frac{-1}{(1+|w_{n,m}|^2)(1+|w_{n+1,m}|^2)}[1-\bar{w}_{n,m}(w_{n+1,m}\Delta_n w_{n,m} - \bar{w}_{n+1,m}\Delta_n w_{n,m})] \quad (6.17)$$
$$\Delta_1 X^{(3)} = \frac{-2}{(1+|w_{n,m}|^2)(1+|w_{n+1,m}|^2)}[w_{n+1,m}\Delta_n w_{n,m} - \bar{w}_{n+1,m}\Delta_n w_{n,m}]$$

and

$$\Delta_2 X^{(1)} = \frac{\mathrm{i}}{(1+|w_{n,m}|^2)(1+|w_{n,m+1}|^2)}[1+w_{n,m}(\bar{w}_{n,m+1}\Delta_m w_{n,m} - w_{n,m+1}\Delta_m \bar{w}_{n,m})]$$
$$\Delta_2 X^{(2)} = \frac{-1}{(1+|w_{n,m}|^2)(1+|w_{n,m+1}|^2)}[1-w_{n,m}(\bar{w}_{n,m+1}\Delta_m w_{n,m} - w_{n,m+1}\Delta_m \bar{w}_{nm})] \quad (6.18)$$
$$\Delta_2 X^{(3)} = \frac{2}{(1+|w_{n,m}|^2)(1+|w_{n,m+1}|^2)}[\bar{w}_{n,m+1}\Delta_m w_{n,m} - w_{n,m+1}\Delta_m \bar{w}_{n,m}].$$

As an example, let us consider the continuous limit for the third component of $\Delta_2 X$. We get $\frac{\Delta_2 X^{(3)}}{\bar{\sigma}} \to \frac{2}{(1+|w|^2)}(\bar{w}\bar{\partial}w - w\bar{\partial}\bar{w})$ when $\bar{\sigma} \to 0$, which is equal to the right-hand side of (2.13) up to a factor i.

The geometrical characterization of a discrete surface in $\mathbb{R}^3$ is carried out in terms of the quadrilateral cross-ratio of four points $(\vec{X}_1, \vec{X}_2, \vec{X}_3, \vec{X}_4)$ [1]. According to [1] we can define the 2×2 matrix Q, given by

$$Q = (X_1 - X_2)(X_2 - X_3)^{-1}(X_3 - X_4)(X_4 - X_1)^{-1} \quad (6.19)$$

where X_i are the matrices corresponding to the four points $\vec{X}_i$. The eigenvalues of Q, q_1 and q_2 are complex functions of the discrete variables *n* and *m*. They provide the *cross-ratio* of the quadrilateral of the four points defined as the vertices of the vectors $(\vec{X}_1, \vec{X}_2, \vec{X}_3, \vec{X}_4)$ (see figure 3). Q is invariant with respect to translations and dilations in $\mathbb{R}^3$; rotations leave the eigenvalues of Q invariant. These properties of q_i make them very valuable for the characterization of discrete surfaces.

To compute the cross-ratio we consider the four points $X_{n,m}$, $X_{n+1,m}$, $X_{n+1,m+1}$ and $X_{n,m+1}$. Taking into account equations (6.11) we can calculate their distance in terms of the projector P and its shifted values. So equation (6.19) can be written as

$$Q = [TP, P](T[SP, P])^{-1}S[TP, P]([SP, P])^{-1}. \quad (6.20)$$

To compute q_i directly from (6.20), we have to express P, given by equation (3.13), in terms of the solution $w_{n,m}$ of the discrete $\mathbf{C}P^1$ equation (5.9). The expression we get is unworkable, due to the complexity of the expressions involved in the computation. A simplification can be obtained, by taking into account that

- an equivalent characterization can be obtained by considering the product of the two eigenvalues of the matrix Q, $q_1 q_2$,
- as one can see from equation (6.20), Q is traceless and thus $q_1 + q_2 = 0$,
- the product of the eigenvalues of a 2×2 matrix is equal to its determinant,
- the determinant of Q can be easily expressed in terms of the determinant of the commutators of P and its shifted values TP, SP, STP

$$\det(Q) = q_1 q_2 = -q_1^2 = \frac{\det([TP, P])\det(S[TP, P])}{\det([SP, P])\det(T[SP, P])}. \quad (6.21)$$

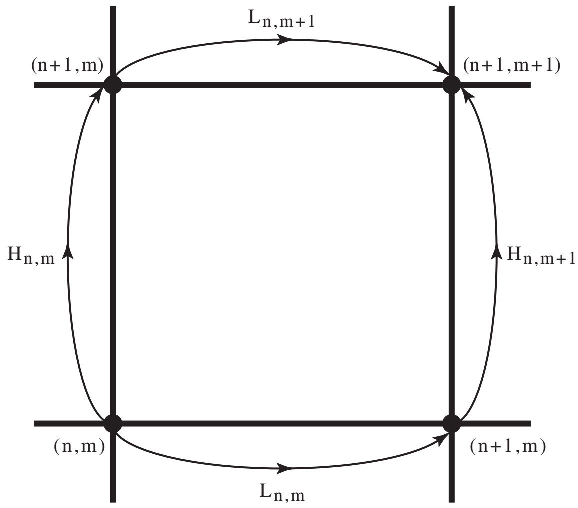

Figure 3. Vertices and edges of a quadrilateral.

The continuous $\mathcal{C}P^1$ equation provides a representation of CMC surfaces. It is natural to assume that also the discrete $\mathcal{C}P^1$ equation provides a representation of CMC surfaces. From [1] we know that discrete CMC surfaces are characterized by a cross-ratio which is the product of two functions, one depending only on n and one on m respectively. Any discrete function $f = f_{n,m} = h_n g_m$ must satisfy the following equation:

$$\Delta_1 \Delta_2 \log(f) = 0. \tag{6.22}$$

Taking into account the properties of the log function, the definitions of Δ_1 and Δ_2 and the expression (6.21) for $\det(Q)$, we are able to show, after a long but trivial *Maple* calculation which involves just the writing of equation (6.22) in terms of $w_{n,m}$ and the substitution of the $\mathcal{C}P^1$ equation (3.15) and its consequences, that q_1 satisfies identically equation (6.22). Thus any discrete surface obtained from solutions of the discrete $\mathcal{C}P^1$ model is a CMC surface.

7. Example

At this point, we would like to illustrate the above considerations with an example of construction of a discrete surface in $\mathbb{R}^3$.

Let us consider a class of discrete surfaces determined by the solutions (5.1) of the difference equation (3.15), which are translationally invariant in the $\bar{\sigma}$ direction. Using expressions (6.11) and (3.13) we can compute the three components of the position vector $X^{(l)}_{n,m}$,

$$\begin{aligned} X^{(1)} &= -\frac{1}{2}\frac{v(n\sigma)+\bar{v}(n\bar{\sigma})}{1+v(n\sigma)\bar{v}(n\bar{\sigma})} \qquad n \in \mathcal{Z} \quad \sigma \in \mathcal{C} \\ X^{(2)} &= \frac{\mathrm{i}}{2}\frac{v(n\sigma)-\bar{v}(n\bar{\sigma})}{1+v(n\sigma)\bar{v}(n\bar{\sigma})} \\ X^{(3)} &= -\frac{1}{1+v(n\sigma)\bar{v}(n\bar{\sigma})}. \end{aligned} \tag{7.1}$$

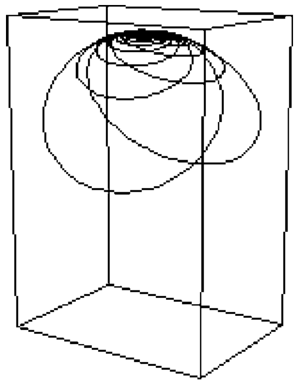

Figure 4. Plotting of the surface obtained by choosing $v_n = n\sigma$ with $\sigma = \cos\phi + \mathrm{i}\sin\phi$. The integer n runs from -25 to 25 and the angle ϕ from 0 to π.

Figure 4 presents a plot of (7.1) for $v_n = n\sigma$ which, for $n \in \mathcal{Z}$ and $\frac{\sigma_R}{\sigma_I} \in \mathbb{R}$, describes a CMC surface. Eliminating v_n we obtain an explicit formula for the discrete CMC surface defined on the lattice

$$(X^{(1)})^2 + (X^{(2)})^2 + (X^{(3)})^2 + X^{(3)} = 0 \tag{7.2}$$

which describes a surface of an ellipsoid. Such a solution has applications in cosmological problems [4] and in the theory of fluid membranes [7].

8. Conclusions and further comments

In this paper, we have shown how to discretize the $\mathcal{C}P^1$ sigma model. We have found that the corresponding second-order difference equations involve the rational dependence on the $\mathcal{C}P^1$ field. From this model we have derived the geometry of the associated discrete surfaces by introducing a complex structure on the lattice. We have shown that the quadrilateral cross-ratio of four points on a surface induced by the discrete $\mathcal{C}P^1$ model can be written as a product of two functions, one of n and one of m only. This implies that any discrete surface associated with a solution of the discrete $\mathcal{C}P^1$ model is a CMC surface.

Work is in progress on the construction of more interesting solutions of the discrete $\mathcal{C}P^1$ model via Darboux and Bäcklund transformations from the translationally invariant solutions presented here and from the new solutions, obtained by studying the $\mathcal{C}P^1$ model's symmetries. Moreover, we are currently looking for a generalization of our results to the case of the discrete $\mathcal{C}P^N$ models [10, 13, 14].

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Appendix A. Useful formulae

Now we list some useful discrete formulae which have facilitated our computations.

A.1. Discrete version of the Leibnitz rule

- the difference of a product of two functions f and g is

$$\Delta_1(f \cdot g) = (\Delta_1 f)g + f(\Delta_1 g) + (\Delta_1 f)(\Delta_1 g). \tag{A.1}$$

In particular, when $f = g$ we get

$$\Delta_1(f^2) = \{\Delta_1 f, f\} + (\Delta_1 f)^2 \tag{A.2}$$

where the brackets { , } denote the anticommutator

$$\{a, b\} = ab + ba.$$

- the difference of the ratio of two functions f and g is

$$\Delta_1\left(\frac{f}{g}\right) = \frac{g(\Delta_1 f) - (\Delta_1 g)f}{g(g + \Delta_1 g)}. \tag{A.3}$$

A.2. Vectors in $\mathbb{R}^3$ and matrices of rank 2

Let us map vectors $\vec{X}$ and $\vec{Y}$ onto two anti-Hermitian 2×2 matrices

$$X = \mathrm{i}X_j\sigma_j \qquad Y = \mathrm{i}Y_k\sigma_k \tag{A.4}$$

where the Pauli matrices σ_i satisfy the following relations

$$[\sigma_i, \sigma_j] = 2\mathrm{i}\epsilon_{ijk}\sigma_k \qquad \{\sigma_i, \sigma_j\} = 2\delta_{ij} \qquad \sigma_i\sigma_j = \mathrm{i}\epsilon_{ijk}\sigma_k + \delta_{ij}. \tag{A.5}$$

The scalar product of two vectors $\vec{X}$ and $\vec{Y}$ in terms of the matrices X and Y is then given by

$$\langle\vec{X}, \vec{Y}\rangle = \tfrac{1}{2}\,\mathrm{tr}\, X^\dagger Y \tag{A.6}$$

since

$$\langle\vec{X}, \vec{Y}\rangle = \tfrac{1}{2}\,\mathrm{tr}\,[(-\mathrm{i}X_j\sigma_j)(\mathrm{i}Y_k\sigma_k)] = \tfrac{1}{2}\,\mathrm{tr}\,(X_jY_k\delta_{jk}) = X_jY_j.$$

The vector product of $\vec{X}$ and $\vec{Y}$ in matrix language is

$$\vec{X} \times \vec{Y} = -\tfrac{1}{2}[X, Y] \tag{A.7}$$

since

$$-\tfrac{1}{2}[X, Y] = \epsilon_{lkj}X_lY_k\sigma_j = \vec{X} \times \vec{Y}.$$

If $X^\dagger = -X$ then the norm

$$\|X\|^2 = \langle\vec{X}, \vec{X}\rangle = \tfrac{1}{2}\,\mathrm{tr}\, X^\dagger X = X_jX_j \geqslant 0 \tag{A.8}$$

is positively defined. If $Z = -\frac{1}{2}[X, Y]$ then $Z^\dagger = Z$ and its norm is

$$\|Z\|^2 = \tfrac{1}{4}\,\mathrm{tr}\,[XYXY - X^2Y^2]. \tag{A.9}$$

The unit vector $\vec{Z}$, in terms of matrices X and Y, is

$$\vec{Z} = \frac{-2[X, Y]}{\mathrm{tr}\,[XYXY - X^2Y^2]}. \tag{A.10}$$

Appendix B. Representation of the matrices *K* and *M* in terms of a projection matrix *P*

Let us look for solutions of the difference equations (3.6), (3.7). The forms of the matrices K and M in terms of P are given by equation (3.9). Substituting equation (3.9) in equation (3.6) we get

$$\begin{aligned}[][T^{\alpha_0} S^{\beta_0+1} \Delta_2 P]\Delta_1 P - [T^{\alpha_1} S^{\beta_1} \Delta_1 P][T^{\alpha_2} S^{\beta_2} \Delta_2 P] - [T^{\alpha_1} S^{\beta_1+1} P][T^{\alpha_2} S^{\beta_2} \Delta_1 \Delta_2 P] \\ + [T^{\alpha_0} S^{\beta_0} \Delta_1 \Delta_2 P]P + [T^{\gamma_0} S^{\delta_0} \Delta_1 \Delta_2 P]P - [T^{\gamma_1} S^{\delta_1} \Delta_2 P][T^{\gamma_2} S^{\delta_2} \Delta_1 P] \\ + [T^{\gamma_0+1} S^{\delta_0} \Delta_1 P]\Delta_2 P - [T^{\gamma_1+1} S^{\delta_1} P][T^{\gamma_2} S^{\delta_2 \Delta_2} \Delta_1 P] = 0. \qquad \text{(B.1)}\end{aligned}$$

Let us note that the first four terms appearing in (B.1) correspond to $\Delta_1 K$, while the second four terms correspond to $\Delta_2 M$. We require that equation (B.1) contains no first-order difference terms. This can happen only if

$$\begin{aligned} \alpha_0 = \gamma_1 \qquad \beta_0 + 1 = \delta_1 \qquad \alpha_1 = \gamma_0 + 1 \\ \beta_1 = \delta_0 \qquad \alpha_2 = \beta_2 = 0 \qquad \gamma_2 = \delta_2 = 0. \end{aligned} \qquad \text{(B.2)}$$

Hence equation (B.1) becomes

$$\begin{aligned}[][T^{\alpha_0} S^{\beta_0} \Delta_2 \Delta_1 P]P - [T^{\gamma_0+1} S^{\delta_0+1} P]\Delta_2 \Delta_1 P \\ + [T^{\gamma_0} S^{\delta_0} \Delta_2 \Delta_1 P]P - [T^{\alpha_0+1} S^{\beta_0+1} P]\Delta_2 \Delta_1 P = 0. \qquad \text{(B.3)}\end{aligned}$$

Without loss of generality one can choose $\alpha_0 = \beta_0 = \gamma_0 = \delta_0 = 0$ and equation (B.3) takes the form (3.11). Equation (3.7) is identically satisfied by virtue of equation (3.10).

Appendix C. A discrete analogue of the integration of total differentials

At this point a clear analogy between our approach and integration of a total differential for a discrete case can be seen, since the form of the compatibility conditions (6.2), at which we arrive here, resembles the integration of a total differential along a closed contour.

In our approach, the construction of discrete surfaces requires solving systems (6.1) and (6.2) for the position vector $X_{n,m}$. Let us denote the right-hand sides of these systems respectively by

$$\Delta_2 X = H \qquad \text{(C.1)}$$

$$\Delta_1 X = L. \qquad \text{(C.2)}$$

So we have

$$\Delta_1 H - \Delta_2 L = 0. \qquad \text{(C.3)}$$

From (C.1) for $m \geqslant 0$ we get

$$X_{n,m} = X_n^{(0)} + \sum_{m'=0}^{m-1} H_{n,m'}. \qquad \text{(C.4)}$$

Substituting this result into (C.2) and using equation (C.3) we obtain an equation for just the initial condition $X_n^{(0)}$

$$X_{n+1}^{(0)} = X_n^{(0)} + L_{n,0}. \qquad \text{(C.5)}$$

Solving equation (C.5) for $n \geqslant 0$ we get

$$X_{n,m} = A_0 + \sum_{n'=0}^{n-1} L_{n'0} + \sum_{m'=0}^{m-1} H_{nm'} \qquad n \geqslant 0 \quad m \geqslant 0 \qquad \text{(C.6)}$$

where A_0 is an integration constant (and where the standard rule applies, saying that if the upper index is lower than the bottom index then the sum is zero). In a similar fashion, for $(n \leqslant 0, m \geqslant 0)$, $(n \geqslant 0, m \leqslant 0)$ and $(n \leqslant 0, m \leqslant 0)$ we get

$$X_{n,m} = A_0 - \sum_{n'=n}^{-1} L_{n',0} + \sum_{m'=0}^{m-1} H_{n,m'} \qquad n \leqslant 0 \quad m \geqslant 0 \tag{C.7}$$

$$X_{n,m} = A_0 - \sum_{n'=n}^{-1} L_{n',0} - \sum_{m'=m}^{-1} H_{n,m'} \qquad n \geqslant 0 \quad m \leqslant 0 \tag{C.8}$$

$$X_{n,m} = A_0 - \sum_{n'=n}^{-1} L_{n',0} + \sum_{m'=m}^{-1} H_{n,m'} \qquad n \leqslant 0 \quad m \leqslant 0. \tag{C.9}$$

Let us note that $X_{n,m}$ is continuous on the whole $(z, \bar{z})$ plane. In equations (C.6)–(C.9) the summation is done at first along the z-axis and then along the axis parallel to the $\bar{z}$-axis and passing through the point $(n, 0)$. It is easy to show, taking into account equation (C.3), that one obtains the same value for $X_{n,m}$ for any path in the $(z, \bar{z})$. In fact, if we calculate the values of $X_{n,m}$ from equations (C.6)–(C.9) along the generic closed contour given in figure 3, we have

$$\begin{array}{ll} X_{n,m+1} - X_{n,m} = H_{n,m} & X_{n+1,m+1} - X_{n,m+1} = L_{n,m+1} \\ X_{n+1,m} - X_{n+1,m+1} = -H_{n+1,m} & X_{n,m} - X_{n+1,m} = -L_{n,m}. \end{array} \tag{C.10}$$

The sum of equations (C.10), which corresponds to a path along the closed elementary circuit in figure 3, is identically satisfied, by virtue of equation (C.3). So, it can immediately be proved, by the standard procedures of complex analysis, that the summation along any closed contour will also be zero. This is the discrete analogue of the Cauchy–Goursat theorem [15].

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